Localised Lifting Lemmas

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The fibration

$$\mathrm{Diff}_\partial \to \widetilde{\mathrm{Diff}_\partial} \to \widetilde{\mathrm{Diff}_\partial}/\mathrm{Diff}_\partial$$

produces the LES

$$\cdots \pi_i \mathrm{Diff}_\partial \to \pi_i \widetilde{\mathrm{Diff}_\partial} \to \pi_i \widetilde{\mathrm{Diff}_\partial}/\mathrm{Diff}_\partial \to \cdots$$

Let p be a prime and denote $\mathbb{Z}_{(p)}$ the localisation of $\mathbb{Z}$ at (p) and denote $G_{(p)} = G \otimes \mathbb{Z}_{(p)}$ for some group G.

Lemma. *$\mathbb{Z}_{(p)}$ is a flat $\mathbb{Z}$ module.*

Thus we can tensor our LES to produce another LES

$$\cdots \pi_i(\mathrm{Diff}_\partial)_{(p)} \to \pi_i(\widetilde{\mathrm{Diff}_\partial})_{(p)} \to \pi_i(\widetilde{\mathrm{Diff}_\partial}/\mathrm{Diff}_\partial)_{(p)} \to \cdots$$

Because $\pi_i(\widetilde{\mathrm{Diff}_\partial})_{(p)}$ is finite, there are no homomorphisms from it to $\mathbb{Z}_{(p)}$. Thus it is sufficient for $\pi_i(\widetilde{\mathrm{Diff}_\partial}/\mathrm{Diff}_\partial)$ to be free with some q torsion coprime to p for the map $\pi_i(\widetilde{\mathrm{Diff}_\partial})_{(p)} \to \pi_i(\widetilde{\mathrm{Diff}_\partial}/\mathrm{Diff}_\partial)_{(p)}$ to be zero (and therefore to unblock p torsion elements).

Now we claim

Lemma. *For an odd prime and in the stable range,*

$$\textit{finite part of } \pi_i(\widetilde{\mathrm{Diff}_\partial}/\mathrm{Diff}_\partial)_{(p)} \cong \textit{finite part of } \pi_{i-1}(\mathscr{C})_{(p)}$$

We actually know more as well about the free part. We know that the free part can only come from the same free part of that concordance group but that depending on the action of $\mathbb{Z}_2$ it may collapse to a $\mathbb{Z}_2$.

Therefore we can put this back in our localised LES to get

$$\cdots \pi_i(\mathrm{Diff}_\partial)_{(p)} \to \pi_i(\widetilde{\mathrm{Diff}_\partial})_{(p)} \to \text{a subgroup of } \pi_{i-1}\mathscr{C}_{(p)} \to \cdots$$

The point is then that if the prime factors of the concordance group differ from Θ_n then I can pull back to where they differ (the horrozontal map will be zero). From which we can immediately conclude

Lemma (A)**.** *For an odd prime p in the stable range, if $\pi_{i-1}\mathscr{C}(D^n)_{(p)} = 0, \mathbb{Z}_{(p)}$ then all elements of order p in Θ_{n+i+1} lift to $\pi_i \mathrm{Diff}_\partial(D^n)$ in the Gromoll filtration. In other words $(\Theta_{n+i+1})_{(p)} \cong (\Gamma^{n+i+1}_{(n)})_{(p)}$.*

Lemma (B)**.** *For an odd prime p in the stable range, if $\pi_i\mathscr{C}(D^n)_{(p)} = 0$ then $\pi_i \mathrm{Diff}_\partial(D^n)_{(p)} \cong (\Gamma^{n+i+1}_{(n)})_{(p)}$.*

Lemma (C)**.** *If $(\Theta_{n+i+1})_{(p)} = 0$ then a subgroup of $\pi_i\mathscr{C}_{(p)}$ surjects onto $\pi_i \mathrm{Diff}_\partial(D^n)_{(p)}$ and a subgroup of $\pi_{i-1}\mathscr{C}_{(p)}$ injects into $\pi_{i-1} \mathrm{Diff}_\partial(D^n)_{(p)}$.*

In other words in this case we can conclude that $\pi_i \mathrm{Diff}_\partial(D^n)_{(p)}$ is an extension of a subgroup of $\pi_i\mathscr{C}_{(p)}$ by the cokernel of the inclusion a subgroup of $\pi_i\mathscr{C}_{(p)} \to \pi_i \mathrm{Diff}_\partial(D^n)_{(p)}$ which we denote CK below.

Now let us consider some cases, if we care about $\pi_i \mathrm{Diff}_\partial(D^n)_{(p)}$ then the relevant LES we have is

$$(\Theta_{n+i+2})_{(p)} \to \text{a subgroup of } \pi_i\mathscr{C}_{(p)} \to \pi_i \mathrm{Diff}_\partial(D^n)_{(p)} \to (\Theta_{n+i+1})_{(p)} \to \text{a subgroup of } \pi_{i-1}\mathscr{C}_{(p)}$$

now there are 16 cases that we tabulate:

$(\Theta_{n+i+2})_{(p)} = 0$	$\pi_i\mathscr{C}(D^n)_{(p)} = 0$	$(\Theta_{n+i+1})_{(p)} = 0$	$\pi_{i-1}\mathscr{C}(D^n)_{(p)} = 0$	Conclusion about $\pi_i \mathrm{Diff}_\partial(D^n)_{(p)}$
0	0	0	0	Nothing
0	0	0	1	Lem A: $\twoheadrightarrow (\Theta_{n+i+1})_{(p)}$
0	0	1	0	a subgroup of $\pi_i\mathscr{C}_{(p)} \twoheadrightarrow$
0	0	1	1	Lem A
0	1	0	0	Lem B $:\cong (\Gamma^{n+i+1}_{(n)})_{(p)}$
0	1	0	1	$\cong (\Theta_{n+i+1})_{(p)}$
0	1	1	0	$= 0$
0	1	1	1	$= 0$
1	0	0	0	Lem C : a subgroup of $\pi_i\mathscr{C}_{(p)} \hookrightarrow$
1	0	0	1	$\mathrm{Ext}((\Theta_{n+i+1})_{(p)}, \text{a subgroup of } \pi_i\mathscr{C}_{(p)})$
1	0	1	0	$\cong$ a subgroup of $\pi_i\mathscr{C}_{(p)}$
1	0	1	1	$\cong$ a subgroup of $\pi_i\mathscr{C}_{(p)}$
1	1	0	0	Lem B $:\cong (\Gamma^{n+i+1}_{(n)})_{(p)}$
1	1	0	1	$\cong (\Theta_{n+i+1})_{(p)}$
1	1	1	0	$= 0$
1	1	1	1	$= 0$

We can see that the cases in which we cannot determine the group up to isomorphism are when $0000, 0001, 0010, 0011, 1000, 1001$ (six out of ten, not bad). So we would like a formula or something that relates the prime factors so that I know when I fall into those cases. A formula for precisely when $\big((\Theta_{n+i+2})_{(p)} \neq 0 \text{ and } \pi_i\mathscr{C}(D^n)_{(p)} \neq 0\big)$ would resolve the first four and a formula for precisely when $\big((\Theta_{n+i+1})_{(p)} \neq 0 \text{ and } \pi_i\mathscr{C}(D^n)_{(p)} \neq 0\big)$ would resolve the last two. Unlikely to get any such formula, but the point is that numerically it is rarely the case that the prime factors agree (at least for the bP part).

So now I need to find a relation between the prime factors of Θ_n and concordance.

1 E^2 page.

First we remark that from [Igu88] and [WJR] we know that stably the homotopy groups of concordance spaces are given by those of the smooth Whitehead spectrum (with a shift), by Waldhausens splitting theorem we have that $A(*) \cong \mathbb{S} \vee \mathrm{Wh}(*)$ and by [Dwy80, Prop 1.2] we have that $\pi_* A(*)$ are all finitely generated. Thus we can conclude that the homotopy groups of concordance spaces are stably finitely genearted abelian groups.

Now according to [Hat78] his spectral sequence for M has E^2 page given by The p here has nothing to do with our prime, fix.

$$E^2_{pq} = H_p(\mathbb{Z}_2; \pi_q \mathscr{C}(M))$$

in the stable range and moreover it converges to

$$E^2_{pq} \implies \pi_{p+q+1}\widetilde{\mathrm{Diff}}_\partial(M)/\mathrm{Diff}_\partial(M).$$

First we will compute the factors of the groups on the E^2 page.

Because the homotopy groups of concordance are finitely generated abelian groups we can write $\pi_q\mathscr{C}(M) = \mathbb{Z}^{n_q} \oplus O_q \oplus \Sigma_q$ where O_q is a finite group of odd torsion, Σ_q is a finite group of even torsion and it is known that $n_q = 0, 1$.

Lemma. $H_p(G; A \oplus B) \cong H_p(G; A) \oplus H_p(G; B)$

Proof. Group homology is an additive functor in the second variable? Homology is at least (of a chain complex). Or apply the singluar construction to BG and see that it distributes.

So we now compute $H_p(\mathbb{Z}_2; \mathbb{Z}), H_p(\mathbb{Z}_2; O_q)$ and $H_p(\mathbb{Z}_2; \Sigma_q)$ seperately using [Wei94, Thm 6.2.2].

1.1 Odd Torsion

Without loss of generality we can assume that $O_q = \mathbb{Z}_{p^n}$. First we claim that O_q has no non-trivial $\mathbb{Z}_2$ module structure.

Proof.

Using the notation there we have that $N = 1 + \sigma$ (in the group ring of $\mathbb{Z}_2$) where σ is a generator of $\mathbb{Z}_2$. Therefore we can use the formula from Weibel with the trivial action to see that

$$H_p(\mathbb{Z}_2; O_q) \cong \begin{cases} O_q/0O_q, & p = 0 \\ O^q/(1+\sigma)O_q, & p \text{ odd} \\ \{a : 2a = 0\}/(\sigma - 1)O_q, & p \text{ even} \end{cases}$$

Becuase both σ acts trivially we see that $(1 + \sigma)O_q = 2O_q$.

Lemma. *If* $\gcd(p, q) = 1$ *then*

$$\times q : \mathbb{Z}_{p^n} \to \mathbb{Z}_{p^n}$$

defines an isomorphism.

Proof.

From this it immediately follows that $2O_q = O_q$ and if $2a = 0$ then $a = 0$. Thus we conclude that

$$H_p(\mathbb{Z}_2; O_q) \cong \begin{cases} O_q, & p = 0 \\ 0, & p \neq 0 \end{cases}$$

1.2 Free Part

What we claim here is that it either stays free or becomes even torsion (or zero).

Lemma. *There are exactly two actions of* $\mathbb{Z}_2$ *on* $\mathbb{Z}$ *, given by the trivial action and the sign action.*

Proof.

Weibel computes the trivial action to be

$$H_p(\mathbb{Z}_2;\mathbb{Z}_{\text{triv}}) = \begin{cases} \mathbb{Z}, & p = 0 \\ \mathbb{Z}_2, & p \text{ odd} \\ 0, & p \text{ even} \end{cases}$$

While the sign action leads to

$$H_p(\mathbb{Z}_2;\mathbb{Z}_{\text{sgn}}) = \begin{cases} \mathbb{Z}/(-2\mathbb{Z}), & p = 0 \\ 0, & p \text{ odd} \\ 0, & p \text{ even} \end{cases}$$

since there are no non-trivial fixed points (second line) and nothing non-zero in $\mathbb{Z}$ will satisfy $-2a = 0$ (third line).

1.3 Even Torsion

Without loss of generality we can assume that $\Sigma_q = \mathbb{Z}_{2^n}$. All we want to show here is that it stays even torsion. This is evident from Weibels formula for $H_p(\mathbb{Z}_2; A)$ as all the groups are subquotients of A.

2 Concluding

Now that we have the E^2 page we can argue what will happen on the E^∞ page. Summarising what we have shown we have

$$E^2_{pq} = \begin{cases} \begin{cases} \mathbb{Z}_2, \\ \mathbb{Z}^{n_q} \end{cases} \oplus O_q \oplus \mathbb{Z}_{2^j}, & p = 0 \\ \begin{cases} 0, \\ \mathbb{Z}_2 \end{cases} \oplus 0 \oplus \mathbb{Z}_{2^j}, & p \text{ odd} \\ 0 \oplus 0 \oplus \mathbb{Z}_{2^j}, & p \text{ even} \end{cases}$$

Thus free groups and odd torsion groups will appear only when $p = 0$, that is the first column of the spectral sequence. From this we can conclude that they will recive only maps from two-torsion groups. These maps must be zero. Thus the free groups and odd torsion groups on E^2 recive only zero maps (and have only zero maps out as they are the first column in a first quadrant SS) and therefore are stable. Thus we can see that the odd torsion and free groups on E^∞ is *equal* to those on E^2, which we have computed and shown to be $\begin{cases} \pi_{i-1}(\mathscr{C})_{(p)}, \\ \text{The finite part of } \pi_{i-1}(\mathscr{C})_{(p)} \end{cases}$.

Now the $p + q + 1 = i$ diagonal on the E^∞ page defines the group $\pi_i\widetilde{\text{Diff}_\partial}/\text{Diff}_\partial$ via successive extensions that we must now analyse. Becuase we are homologically graded the extension problem begins with the group in $E^\infty_{0,i-1}$ which we have shown is a subgroup of $\mathbb{Z} \oplus O_{i-1} \oplus \tau$ where τ is some even torsion group (it will correspond to the even torsion part of the concordance groups and the quotient of the free part). This group will then be successively extended by even torsion groups. The first extension to deal with is of the form

$$0 \to \mathbb{Z} \oplus O \oplus \Sigma \to G \to E^\infty_{1,i-2} \to 0$$

which will be classified by some class in $\text{Ext}(E^\infty_{1,i-2}, \mathbb{Z} \oplus O \oplus \Sigma)$.

Lemma. *Extension of direct sum is direct sum of extension? Not true need to deal with the extensions directly. Then extension of free things by finite cyclic things are either free or free plus a cyclic thing? Extensions of finite things are split if they are coprime?*

Lemma. $\mathrm{Ext}(A, B \oplus C) \cong \mathrm{Ext}(A, B) \oplus \mathrm{Ext}(A, C)$ *and* $\mathrm{Ext}(B \oplus C, A) \cong \mathrm{Ext}(B, A) \oplus \mathrm{Ext}(C, A)$. *(Wikipedia). [Wei94, Prop 3.3.4]*

Lemma. *If* $A = \mathbb{Z}_{2^n}$ *and* $O = \mathbb{Z}_{p^m}$ *then*

$$\mathrm{Ext}(A, \mathbb{Z}) \cong A^* \cong A$$

where $A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ *is the Pontryagin dual [Wei94, Example 3.3.3].*

$$\mathrm{Ext}(A, O) \cong 0$$

by [Wei94, Calc 3.3.2].

Lemma. *If we have an extension* $A \oplus B \to G \to E$ *and* $\mathrm{Ext}(E, A) = 0$ *then* $G \cong A \oplus H$.

Thus we can see that because all the $E^\infty_{j,i-1-j}, j \geq 1$ are two torsion each extension preserves O as a summand. The extensions of $\mathbb{Z}$ by say $\mathbb{Z}_4$ are given by $\mathbb{Z}$, $\mathbb{Z}_4 \oplus \mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}_2$. This suggests that the free summand will also be preserved. My only concern is that it may interact with the other even torsion part extension.

Lemma. *Claim that for an extension of* $\mathbb{Z}$ *by a finite group the rank is preserved. Claim that extending any group by an even torsion group does not introduce any new odd torsion.*

Given this we have shown that the odd torsion and free rank that is present on the E^2 page persists to the E^∞ page and that it assembles via successive extension problems to a group that has the same rank and the same odd torsion summand. Now since we showed that what appeared on the E^2 page as odd torsion was merely the odd torsion in the homotopy groups of concordance and at most their free part then we are done.

3 Numerics

3.1 Regular Primes

For regular primes $\pi_i \mathscr{C}$ is known stably by [**?**] in a large enough range (easy to compute range gets bigger quadratically with the prime, out side of this something is known but still computational challenge (for me)). Unfortunately there is no systematic way of saying when regular primes appear in Θ_n or not; all I can say is that

- The numerator of $B_k/4k$ factor will not have any regular prime factors for $p \geq 2k + 3$.
- The Mersenne number factor is completely up in the air.

So maybe now we go through the bP spheres, look at what regular primes appear and do it like that. Irregular primes will obviously appear but I have no handle on the higher concordance values for them; In principle I can evaluate maybe a few higher than integrally because I know that the irregular primes in (0?) all have magnitude $> 10^7$ and irregular primes dont show up in some of the lower concordance groups.

3.2 Irregular Primes

It will be easy to summarise what can be said in this case, because it will be little. Let $p < 10^7$ be an irregular prime. Return to `https://rileymoriss.github.io/Notes/Research/PhD/Homotopy_groups_of_Concordance.` and see if I can compute them higher for just this p. At this point Im losing faith in the whole with coefficients approach. I think it makes more sense to just tensor the whole situation with $\mathbb{Z}_{(p)}$..... A lot of the same things hold, its just less contrived in a way?¿

Remark. This only gets me stable results.

References

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